# Evidence for the Nonintegrability of a Water Wave Equation in 2+1 Dimensions

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We provide evidence of the nonintegrability of a recently proposed model for water waves in $2+1$ dimensions: we show that under a nonlinear time transformation, a certain reduction of this partial differential equation is mapped to an ordinary differential equation which does not have the Painlevé property. This is in contrast to what happens in the case of the Camassa-Holm equation. Also, and again in contrast to the case of the Camassa-Holm equation, the equation under study fails to admit Dirichlet series solutions. – MSC2000 classification scheme numbers: 35Q51, 35Q58, 37K10.

*Key words:* Nonintegrability; Similarity Reductions.

## 1. Introduction

Completely integrable non-semilinear differential equations, for example as discussed in [1 – 4], have long been of interest, as have their integrable extensions in higher dimensions [5]. For a discussion of integrability in multidimensions, we refer to [6, 7]. Thus, since the derivation of the Camassa-Holm (CH) equation,

$$u_t + \kappa u_x - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \quad (1)$$

both from a mathematical point of view [8] and also in the context of water waves [9], there has been a great deal of interest in the question of its possible generalization to higher dimensions. One such generalization, to $2+1$ dimensions, was obtained in [10], this system being the subject of further analysis in [11 – 13].

Another generalization of the CH equation to $2+1$ dimensions has recently been derived in [14], from physical considerations, “as an example of what is possible”. This equation is therein referred to as the CH counterpart of the two-dimensional Korteweg-de Vries, or Kadomtsev-Petviashvili, equation, and can be written as

$$\begin{aligned} &u_{yy} + (u_t + \kappa u_x - u_{xxt} + 3uu_x \\ &\quad - 2u_x u_{xx} - uu_{xxx})_x = 0. \end{aligned} \quad (2)$$

In [14] it is noted that the possibility of this equation being completely integrable is worth investigating; it is this equation that we will study in the present paper. We provide evidence, amongst other results, that it is in fact nonintegrable.

The layout of the paper is as follows. In Sect. 2 we consider the application of the Weiss-Tabor-Carnevale Painlevé test [15]; we obtain that (2) admits only weak Painlevé expansions. In Sect. 3 we derive several $(1+1)$-dimensional partial differential equations (PDEs) as reductions of (2), using the Lie symmetry approach. In Sect. 4 we analyse a further reduction of one of these PDEs to an ordinary differential equation (ODE), comparing our results to those obtained by the same process for the CH equation itself. In Sect. 5 we consider the construction of formal Dirichlet series solutions of (2), again comparing to the results obtained for the CH equation. Section 6 is devoted to conclusions.

## 2. The Painlevé PDE Test

In this section we consider the singularity analysis of (2). The application of the Painlevé PDE test presents the same difficulties that arise when dealing with the CH equation and also with the alternative $(2+1)$-dimensional generalization of the CH equation introduced in [10]; for the application of the Painlevé PDE test to this latter see [13]. These difficulties can be overcome by including an extra lower order term in the expansion [16]; see (3) and (4) below.

We consider seeking a local Laurent expansion about a noncharacteristic movable singular manifold $\phi = 0$ [15]. We find the leading order behaviour

$$u \sim -\psi_t + u_0 \phi^{(2/3)}, \tag{3}$$

where here we are using Kruskal's ansatz [17], that is, we are taking $u_0$ to be a function of $(y,t)$ only and $\phi(x,y,t) = x + \psi(y,t)$. Here the term $-\psi_t$ is an additional lower order term that corrects the balancing of terms in the leading order analysis. We obtain as resonances $r = -1, 0, \frac{2}{3}, \frac{5}{3}$. The resonance at $r = -1$ is associated with the arbitrariness of the function $\psi$, and that at $r = 0$ to the arbitrariness of the coefficient $u_0$. The remaining resonances indicate that we have to modify the standard Laurent series expansion and consider instead a "weak Painlevé" expansion, or Puiseux series. We therefore seek an expansion of the form

$$u = -\psi_t + \phi^{\frac{2}{3}} \sum_{j=0}^{\infty} u_{\frac{j}{3}} \phi^{\frac{j}{3}} \tag{4}$$

with the coefficients functions of $(y,t)$ and $u_0$ arbitrary. We find that the compatibility conditions at $r = \frac{2}{3}$ and $r = \frac{5}{3}$ are satisfied, and so the expansion (4) contains four arbitrary functions $(\psi, u_0, u_{\frac{2}{3}}, u_{\frac{5}{3}})$ of $(y,t)$, with all other coefficients in the series being determined in terms of these. Thus we conclude that (2) fails the Painlevé PDE test, admitting only a weak Painlevé expansion.

This result, interesting though it may be, does not however allow us to conclude that the non-semilinear PDE (2) is not integrable; the completely integrable Dym equation, for example, exhibits similar behaviour [18]. A further analysis of (2) is therefore needed; this we carry out in Section 4. First we consider the similarity reductions of (2).

## 3. Similarity Reductions

The application of the classical Lie group method [19–21] requires considering a one-parameter Lie group of infinitesimal transformations in the variables $(x,y,t,u)$ given by

$$\begin{aligned} x &\to x + \varepsilon\,\xi_1(x,y,t,u) + O(\varepsilon^2), \\ y &\to y + \varepsilon\,\xi_2(x,y,t,u) + O(\varepsilon^2), \\ t &\to t + \varepsilon\,\tau(x,y,t,u) + O(\varepsilon^2), \\ u &\to u + \varepsilon\,\eta(x,y,t,u) + O(\varepsilon^2), \end{aligned} \tag{5}$$

where $\varepsilon$ is the group parameter. The condition that the above transformation leaves invariant the PDE under consideration yields an overdetermined system of linear equations for the infinitesimals $\xi_1(x,y,t,u)$, $\xi_2(x,y,t,u)$, $\tau(x,y,t,u)$ and $\eta(x,y,t,u)$. The associated Lie algebra consists of vector fields of the form

$$\begin{aligned} \boldsymbol{v} = \xi_1(x,y,t,u)\frac{\partial}{\partial x} + \xi_2(x,y,t,u)\frac{\partial}{\partial y} \\ + \tau(x,y,t,u)\frac{\partial}{\partial t} + \eta(x,y,t,u)\frac{\partial}{\partial u}. \end{aligned} \tag{6}$$

Once the infinitesimal generators have been determined, the symmetry variables for the associated reduction can be found by solving the characteristic equations

$$\begin{aligned} \frac{\mathrm{d}x}{\xi_1(x,y,t,u)} = \frac{\mathrm{d}y}{\xi_2(x,y,t,u)} = \\ \frac{\mathrm{d}t}{\tau(x,y,t,u)} = \frac{\mathrm{d}u}{\eta(x,y,t,u)}. \end{aligned} \tag{7}$$

For (2) the infinitesimals are

$$\xi_1 = -c_1\kappa t + c_2, \tag{8}$$

$$\xi_2 = c_1 y + c_3, \tag{9}$$

$$\tau = 2c_1 t + c_4, \tag{10}$$

$$\eta = -c_1(2u + \kappa), \tag{11}$$

where $c_1, c_2, c_3$ and $c_4$ are arbitrary constants. The above infinitesimal generators provide for (2) different nontrivial symmetry reductions, depending on whether $c_1$ is zero or different from zero. These we now consider in detail.

*Case 1:* $c_1 \neq 0$. In this case we can set $c_1 = 1$ and $c_2 = c_3 = c_4 = 0$ without loss of generality. Solving the characteristic equations (7), we find the similarity reduction

$$u(x,y,t) = \frac{\omega(z,\tau)}{t} - \frac{\kappa}{2}, \quad z = x + \frac{\kappa}{2}t, \quad \tau = \frac{y^2}{t}, \tag{12}$$

under which (2) reduces to

$$\begin{aligned} (4\tau\omega_\tau - 2\omega - \tau\omega_z + \tau\omega_{zzz})_\tau \\ + \left(\frac{3}{2}\omega^2 - \omega\omega_{zz} - \frac{1}{2}\omega_z^2\right)_{zz} = 0. \end{aligned} \tag{13}$$

*Case 2:* $c_1 = 0$. In this case we have different possibilities according to the values of the constants $c_2, c_3$ and $c_4$.

*Subcase 2a):* $c_4 \neq 0$. We can without loss of generality take $c_4 = 1$; the corresponding similarity reduction is

$$u(x,y,t) = \omega(z,\tau), \quad z = x - c_2 t, \quad \tau = y - c_3 t, \quad (14)$$

which yields the PDE

$$\omega_{\tau\tau} - c_3\omega_{z\tau} + (c_2\omega_{zz} + c_3\omega_{z\tau} - \omega\omega_{zz} - \frac{1}{2}\omega_z^2 + \frac{3}{2}\omega^2 + (\kappa - c_2)\omega)_{zz} = 0. \quad (15)$$

*Subcase 2b):* $c_4 = 0$. In this case we can set $c_3 = 1$ without loss of generality, and we obtain the similarity reduction

$$u(x,y,t) = \omega(z,\tau), \quad z = x - c_2 y, \quad \tau = t, \quad (16)$$

which gives the equation

$$\big((c_2^2 + \kappa)\omega_z + \omega_\tau + 3\omega\omega_z - \omega_{zz\tau} - \omega\omega_{zzz} - 2\omega_z\omega_{zz}\big)_z = 0. \quad (17)$$

We note that this last is equivalent to the $x$-derivative of the CH equation (1).

## 4. A Nonlinear Time Transformation

It is here that we provide evidence of the nonintegrability of (2). We consider the travelling-wave reduction

$$\omega(z,\tau) = P(X) - m, \quad X = z + m\tau \quad (18)$$

of (17). The resulting ODE in $P(X)$ can be integrated once to give

$$PP_{XXX} + 2P_X P_{XX} - 3PP_X - \alpha P_X + \beta = 0, \quad (19)$$

where $\alpha = \kappa + c_2^2 - 2m$ and $\beta$ is an arbitrary constant of integration. (In fact, (19) can be integrated further, but we choose not to do so here.) We now follow [22] and make in (19) the transformation

$$P(X) = v(\zeta), \quad \frac{\mathrm{d}X}{\mathrm{d}\zeta} = v(\zeta), \quad (20)$$

which yields

$$v^2 v_{\zeta\zeta\zeta} - 2vv_\zeta v_{\zeta\zeta} + v_\zeta^3 - 3v^4 v_\zeta - \alpha v^3 v_\zeta + \beta v^4 = 0. \quad (21)$$

It is straightforward to show that this ODE has the Painlevé property if and only if $\beta = 0$. First, applying the Ablowitz-Ramani-Segur algorithm [23] using the leading order $v \sim 1/(\zeta - \zeta_0)$, we find as resonances $r = -1, 2, 3$; the compatibility conditions at $r = 2$ and $r = 3$ are satisfied if and only if $\beta = 0$. Thus if $\beta \neq 0$, the general solution of the ODE (21) exhibits logarithmic branching. Second, if $\beta = 0$, the ODE (21) can be integrated to obtain

$$v_{\zeta\zeta} = \frac{1}{2}\frac{v_\zeta^2}{v} + \frac{3}{2}v^3 + \alpha v^2 + Cv, \quad (22)$$

or alternatively

$$v_{\zeta\zeta} = \frac{v_\zeta^2}{v} + v^3 + \frac{1}{2}\alpha v^2 + D, \quad (23)$$

where $C$ and $D$ are constants of integration. These last two ODEs, being special cases of $P_{XXX}$ and $P_{XII}$ respectively [24], have the Painlevé property.

Thus we see that the transformation (20) yields an ODE which does not have the Painlevé property. This is in contrast to the case of the CH equation discussed in [22], where the same transformation was used to map a reduction of this last to $P_{XXX}$. That here we use the same transformation as in [22] is due to the close relationship between the CH equation and the PDE (17) derived from (2); in [22] it is the ODE (19) with $\beta = 0$ that is derived from the CH equation.

Of course, the nonlinear time transformation is closely related (but at the ODE level) to a hodograph transformation. We remark that hodograph transformations are well-established as a tool for mapping completely integrable but non-semilinear PDEs, which do not pass the Painlevé PDE test, onto corresponding completely integrable PDEs which do pass this test. For the case of the CH equation itself, for example, see [25]. Indeed, the use of such nonlinear time / hodograph transformations has been proposed in the classification of ODEs and PDEs [26, 27]. Thus for example in [27], where a hodograph transformation can be applied, it is proposed that a Painlevé analysis be undertaken *after* making such a transformation. In our ODE case, this corresponds to testing (21) for the Painlevé property.

We believe that our results are strongly suggestive of the nonintegrability of the PDE (2). We remark that, since we are interested in obtaining an ODE not having the Painlevé property, indicative of nonintegrability, we do not need to consider further reductions. We now turn to a further difference between (2) and the CH equation.

## 5. Dirichlet Series

In two recent papers [28, 29] we have seen that certain examples of non-semilinear integrable PDEs admit Dirichlet series solutions. In particular, the CH equation admits such a solution of the form

$$u = e^{-x}\sum_{j=0}^{\infty} u_j(t)e^{jx}, \tag{24}$$

where $u_0$, $u_2$ and $u_3$ are arbitrary, as well of course as the series obtained from this last under the discrete symmetry $(x,t)\to(-x,-t)$ of the CH equation. Moreover, it was shown in [28, 29] that for certain classes of equations, the only PDEs admitting Dirichlet series solutions are transformable back onto the only known integrable equations within those classes. Thus it would seem that, for certain classes of equations, there is some kind of relationship between integrability and the admission of Dirichlet series solutions; the nature of this connection is explored in [30].

Here we consider the construction of Dirichlet series solutions for (2). We obtain the possible leading order (or dominant) behaviours $u\sim u_0(y,t)e^{\pm x}$, corresponding to which we find recursion relations of the form

$$j(j\pm 2)^2(j\pm 3)u_j = R_j, \tag{25}$$

where $R_j$ is a function of previous coefficients (these being functions of $(y,t)$), and derivatives thereof. Here we consider the leading order behaviour $u\sim u_0(y,t)e^{-x}$. In order that we have a solution containing four arbitrary functions of $(y,t)$, we need to consider, instead of a series of the form (24), one of the form

$$\begin{aligned} u = e^{-x}\big[&u_0 + u_1e^x + (u_{2,0}+u_{2,1}x+u_{2,2}x^2)e^{2x}\\ &+ (u_{3,0}+u_{3,1}x+u_{3,2}x^2+u_{3,3}x^3)e^{3x}\\ &+ (u_{4,0}+u_{4,1}x+u_{4,2}x^2+u_{4,3}x^3)e^{4x}+\cdots\big], \end{aligned} \tag{26}$$

where all coefficients are functions of $(y,t)$. Substituting this series into (2), we find that $u_0$, $u_{2,0}$, $u_{2,1}$ and $u_{3,0}$ are left arbitrary (we can also obtain a second series, corresponding to the leading order behaviour $u\sim u_0(y,t)e^x$, using the discrete symmetry $(x,t)\to(-x,-t)$ of (2)). The modification with powers of $x$ of our Dirichlet series solution of (2) is in contrast to the case of the integrable CH equation, where no such modification is needed.

The reason for this modification is not due solely to the need to include four arbitrary coefficients (with two arbitrary coefficients corresponding to $j = 2$), but also to the fact that, even in seeking a solution of the form (24) with $u_j(y,t)$, we encounter failed "compatibility conditions" at $j = 2$ and $j = 3$ due to the $u_{yy}$ term in (2).

Thus we see that, with respect to the question of Dirichlet series solutions, (2) differs in its behaviour from the integrable CH equation. In [30] we present a variety of non-integrable PDEs which do not admit Dirichlet series solutions, in contrast to further examples of integrable PDEs, which do admit such solutions.

## 6. Conclusions

We have seen that Johnson's $(2+1)$-dimensional generalization of the CH equation, whilst behaving similarly to the CH equation with respect to the Painlevé PDE test, differs markedly from the CH equation in other respects. In particular we have seen that, if we apply the same nonlinear time transformation as was considered in [22] for the CH equation, we obtain an ODE which does not have the Painlevé property. We believe that this constitutes an indication of nonintegrability. Also, (2) fails to admit Dirichlet series solutions, whereas the CH equation does admit such solutions [28, 29].

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